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THE NECESSARY AND SUFFICIENT CONDITIONS UNDER WHICH TWO LINEAR HOMOGENEOUS DIFFERENTIAL EQUA- TIONS HAVE INTEGRALS IN COMMON

By A. B. PIERCE

IN the theory of the linear homogeneous differential equations, many properties and relations are found which are very closely parallel to those of the ordinary algebraic equations. One of these is the necessary and sufficient condition (as given in determinant form) that two such equations should have integrals in common.

The necessity and sufficiency of this condition have been established by von Escherich* and Heffter,† the latter of whom has also proved the conditions for multiple common integrals.‡ The method used in the present paper for establishing the sufficiency of this condition follows entirely new lines and has many points of principle in common with that for the corresponding algebraic problem as given by Mansion in his *Elements de la theorie des determinants*.§

In §1, I have proved the necessary condition after the method of von Escherich, which is the analogue of Sylvester's dialytic method of elimination as used in the algebraic problem. The proof of the main problem in §2 is divided into two distinct parts: in the first part is the proof of a theorem of considerable generality and value aside from the purposes of this paper, which expresses a sufficient condition in symbolic form; in the second part, after a preliminary definition, and discussion and proofs of the properties of

* *Denkschriften der Wiener Academie*, vol. 46 (1883).

† *Crelle*, vol. 116 (1896) and *Archiv der Mathematik und Physik*, ser. 3, vol. 3 (1902).

‡ *Archiv der Mathematik und Physik*, l. c.

§ Mansion: *Elements de la theorie des determinants*, sixth edition, Paris, 1900.

matrices involved, the sufficiency of the condition of §1 is proved on the basis of the above mentioned theorem. The form of proof here given was adopted to avoid the consideration of certain special cases which would otherwise require separate discussion, and also to make the transition to the cases of §3 simple and direct. In §3, I have proved necessary and sufficient conditions for the case of several independent integrals in common.

It is also to be noticed that the discussion here given, gives the results for the algebraic problem by using the differential equations with constant coefficients, as will be pointed out more in detail below.

We shall indicate by the symbol A the linear homogeneous differential operator

$$a_0 \frac{d^a}{dx^a} + a_1 \frac{d^{a-1}}{dx^{a-1}} + \cdots + a_{a-1} \frac{d}{dx} + a_a$$

of order a , in which the coefficients are functions of x ; and also by Ay the result obtained by applying the operator A upon the quantity y , namely:

$$a_0 \frac{d^a y}{dx^a} + a_1 \frac{d^{a-1} y}{dx^{a-1}} + \cdots + a_{a-1} \frac{dy}{dx} + a_a y.$$

It should be noticed that we assume that the function a_0 is not identically zero. Similarly B, R, S are linear homogeneous differential operators of orders b, r, s respectively and whose coefficients, also functions of x , may be indicated by β, ρ, σ respectively; with subscripts to correspond. Also RAy shall mean that A is first applied to y and then R applied to the result; similarly for SB_y .

Throughout this paper, we shall take $a \geq b$.

1. Now since any function which satisfies $Ay = 0$ will also satisfy $\frac{d}{dx} Ay = 0$, by differentiating Ay successively with respect to x and each time placing the result equal to zero, and similarly for By , we obtain a system of equations which is satisfied by every integral common to the two differential equations $Ay = 0$ and $By = 0$. If we differentiate the first $(b - 1)$ times, the second $(a - 1)$ times and rearrange, we obtain a system of $(a + b)$ equations which must be satisfied by every function which satisfies both of the given equations. These equations will be linear and homogeneous in y and its first

2. LEMMA: *If $A(y)$ and $B(y)$ be two linear homogeneous differential expressions such that we can find two linear homogeneous differential operators R and S for which $RAy \equiv SBy$ and $s < a$, then the two differential equations $Ay = 0$ and $By = 0$ have at least one integral, other than zero, in common.*

From the condition of identity in the hypothesis, $r + a = s + b$ and therefore $r < b$ if $s < a$.

Let $y_{a,1}, y_{a,2}, \dots, y_{a,a}$ be a fundamental system of integrals of the differential equation $Ay = 0$.

Since $Ay_{a,i} = 0$, and therefore $RAy_{a,i} = 0$, for $i = 1, 2, \dots, a$, our identity gives $SBy_{a,i} = 0$ for the same system of values of i . As a result, we have the following two possibilities: either (1) one or more of the functions $y_{a,i}$ is an integral of the equation $By = 0$; or (2) $y_{a,i}$, when substituted for y in the expression By , always gives an integral, not identically zero, of the differential equation $Sy = 0$.

The first of these possibilities establishes the theorem stated without further discussion, so we turn to the second alternative.

Let us indicate the results of the substitution as follows:

$$By_{a,1} = y_{s,1}, \quad By_{a,2} = y_{s,2}, \quad \dots, \quad By_{a,s} = y_{s,s}, \\ By_{a,s+1} = y_{s,s+1}, \quad \dots, \quad By_{a,a} = y_{s,a}.$$

Of these functions $y_{s,k}$ not more than s can be linearly independent, so we will consider that our integrals $y_{a1}, y_{a2}, \dots, y_{aa}$ have been so arranged that all of the functions $y_{s,k}$ which are independent are included in the first s of the a given above. Then we can write

$$y_{s,s+l} = \lambda_{1,l} y_{s,1} + \lambda_{2,l} y_{s,2} + \dots + \lambda_{s,l} y_{s,s}$$

for all values of l from 1 to $(a-s)$ inclusive, where the quantities λ are constants and not all zero.

Let us now form a solution $Y_{a,s+l}$ of the equation $Ay = 0$ in the same way out of $y_{a,1}, y_{a,2}, \dots, y_{a,s}$, using the same constants λ , as follows:

$$Y_{a,s+l} = \lambda_{1,l} y_{a,1} + \lambda_{2,l} y_{a,2} + \dots + \lambda_{s,l} y_{a,s}.$$

Then $BY_{a,s+l} = y_{s,s+l}$; but we also have $By_{a,s+l} = y_{s,s+l}$. By subtraction, we obtain

$$B(Y_{a,s+l} - y_{a,s+l}) = 0.$$

That is, $(Y_{a,s+l} - y_{a,s+l})$ is an integral of the equation $By = 0$. But it is an integral of the equation $Ay = 0$ since it is a linear combination, with

constant coefficients, of integrals of this equation. Moreover $Y_{a,s+l} - y_{a,s+l}$ is not identically zero since the functions $y_{a,l}$ are linearly independent. Therefore the theorem is proved.

We have also the following corollary:

COROLLARY. *The differential equations $Ay = 0$ and $By = 0$ have at least $(a - s)$ independent integrals in common.*

$Y_{a,s+l}$ contains only the integrals $y_{a,1}, y_{a,2}, \dots, y_{a,s}$, and the integrals $y_{a,s+l}$ are independent integrals for all the different values of l from 1 to $(a - s)$ inclusive. Therefore the expressions $Y_{a,s+l} - y_{a,s+l}$ are linearly independent of each other for the above set of values for l . But they are all integrals of the two equations. Therefore our statement is proved.

In proving the sufficiency of the condition $\Delta(x) \equiv 0$ by means of the lemma just proved, certain complications arise on account of the great number of ways in which this determinant may vanish: for example, by the vanishing of all its minors down to a certain definite order; or the vanishing of all the minors which enter matrices chosen according to certain laws, matrices to which I have given the special name, *submatrices* of the various orders, and which are defined as follows:

DEFINITION: The n^{th} submatrix of the determinant $\Delta(x)$ is obtained by cutting off the first n columns, the first n rows, and the last n rows.

We now prove three properties of these submatrices which are fundamental in our discussion.

PROPERTY I: If the n^{th} submatrix vanish, and $n < b$, then $\Delta(x)$ vanishes.

For the matrix obtained by cutting off the first n and also the last n rows of the determinant $\Delta(x)$ vanishes since the columns of this matrix in which all the elements are not zero are the columns of the n^{th} submatrix. Now expanding Δ according to its minors found in this matrix, we find that every term of this expansion has one factor zero and therefore $\Delta(x)$ must be zero.

PROPERTY II: If the $(n + 1)^{\text{st}}$ submatrix does not vanish, not all the first minors of the n^{th} submatrix can vanish.

Let D be a determinant of $(n + 1)^{\text{st}}$ submatrix which does not vanish. Construct the determinant of order one higher than D in which the first element of the first column is a_0 and all the other elements of that column are zeros and D is the minor of a_0 , and also all the other elements of the first row are the elements of the first row of the n^{th} submatrix which occur in the columns with the elements of the columns of D . Then this new determinant is a first minor

of the n^{th} submatrix and being equal to $a_0 D$ is not identically zero: thus proving our property.

PROPERTY III: In $\Delta(x)$, there is always a submatrix which does not vanish if $Ay \neq f(x) \cdot By$.

If $a > b$, the b^{th} submatrix is such a one; its first determinant being $\beta_0^{a-b} \neq 0$. If $a = b$, the $(b-1)^{\text{st}}$ is such. Here the matrix has two rows and if it vanished we should have the relation

$$a_0 : \beta_0 = a_1 : \beta_1 = \dots = a_k : \beta_k = \dots,$$

which is excluded by the hypothesis.

We now take up our principal theorem.

THEOREM: If $Ay = 0$ and $By = 0$ be two linear homogeneous differential equations of orders a and b whose coefficients make $\Delta(x)$ vanish identically, then $Ay = 0$ and $By = 0$ have at least one integral in common, which is not identically zero.

We may assume that the first members of the differential equations are not proportional to each other as otherwise these equations would obviously have all their solutions in common. By Property III there exists therefore a submatrix of $\Delta(x)$ which does not vanish. Let the $(n+1)^{\text{st}}$ submatrix be the first one which does not vanish. It follows from the proof of Property III that $n < b$, or if $a = b$ that $n < b - 1$. Then, if $h = a + b - 2n$, it has $(h-2)$ rows and $(h+n-1)$ columns, and the n^{th} submatrix has h rows and $(h+n)$ columns. Writing the n^{th} submatrix, we have

$$\left\| \begin{array}{cccccccc} a_0, & a_1 + (b-n-1)a'_0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_a^{b-n-1} \\ 0, & a_0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_a^{b-n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \cdot & \cdot & \cdot & \cdot & 0, & a_0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_a \\ 0, & 0, & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_0, & \beta_1, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & \beta_0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_b^{a-n-2} \\ \beta_0, & \beta_1 + (a-n-1)\beta'_0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_b^{a-n-1} \end{array} \right\| \equiv 0$$

We shall first assume that the determinants of order $(h-1)$ given by the first $(h-1)$ columns are *not* all zero.

Now form all the determinants of order h in which the first $(h-1)$

columns are the first $(h-1)$ columns of the n^{th} submatrix and the h^{th} column is successively one of the remaining $(n+1)$ columns. In each of these determinants multiply the elements of the last column by one of the factors $\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \dots, \frac{dy}{dx}, y$, according as they belong to the $h^{\text{th}}, (h+1)^{\text{st}}, \dots, (h+n-1)^{\text{st}}, (h+n)^{\text{th}}$, column of the submatrix respectively; and then add them all together. We thus obtain a new determinant in which every element of the last column is a linear homogeneous differential expression of order n . This new determinant vanishes because by hypothesis each of the determinants which we added together to form it vanished.

In the new determinant, we multiply the successive columns beginning with the first and including the $(h-1)^{\text{st}}$ by the factors $\frac{d^{h+n-1}y}{dx^{h+n-1}}, \frac{d^{h+n-2}y}{dx^{h+n-2}}, \dots, \frac{d^{n+1}y}{dx^{n+1}}$ respectively and add to the last column. Placing $r = (b-n-1)$ and $s = (a-n-1)$, the relation can then be rewritten as follows:

$$\left| \begin{array}{cccccccccccc} a_0, & a_1 + ra'_0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{dr}{dx^r}Ay \\ 0, & a_0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{dr-1}{dx^{r-1}}Ay \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \cdot & \cdot & \cdot, & 0, a_0, (a_1 + a'_0), & \cdot & \cdot & \cdot & \cdot & \frac{d}{dx}Ay \\ 0, & 0, & \cdot & \cdot & \cdot & 0, a_0, & \cdot & \cdot & \cdot & \cdot & \cdot & Ay \\ 0, & 0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot, & 0, \beta_0, & \cdot & \cdot & By \\ 0, & 0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot, & \beta_0, (\beta_1 + \beta_0), & \cdot & \cdot & \frac{d}{dx}By \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & \beta_0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{d^{s-1}}{dx^{s-1}}By \\ \beta_0, \beta_1 + s\beta'_0, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{d^s}{dx^s}By \end{array} \right| \equiv 0.$$

If we expand this determinant according to the elements of the last column, we have the form

$$\begin{aligned} & \rho_0 \frac{d^r}{dx^r} Ay + \rho_1 \frac{d^{r-1}}{dx^{r-1}} Ay + \cdots + \rho_{r-1} \frac{d}{dx} Ay + \rho_r Ay + \cdots \\ & - \left\{ \sigma_s By + \sigma_{s-1} \frac{d}{dx} By + \cdots + \sigma_1 \frac{d^{s-1}}{dx^{s-1}} By + \sigma_0 \frac{d^s}{dx^s} By \right\} \equiv 0. \end{aligned}$$

In this scheme, we notice, that, starting from the right, the elements of the first column of the submatrix are multiplied by y , those of the second column by $\frac{dy}{dx}$, those of the third by $\frac{d^2y}{dx^2}$, and so on.

It may happen that the particular $(h-1)$ columns, which we have picked out of the submatrix do not fulfil the hypothesis explicitly made concerning them and that in fact the quantities ρ_i and σ_j all vanish identically. But from Property II, we see that we can choose $(h-1)$ columns which do fulfil this hypothesis. Then we may proceed in a similar manner, always associating the elements of every column with the derivative of y of proper order.

Therefore we always have an expansion like that given above in which the quantities ρ_i and σ_j do not all vanish. Indeed from Property II, we see that the $(h-1)$ columns can be chosen so that neither ρ_0 nor σ_0 is zero.

Writing

$$\rho_0 \frac{d^r}{dx^r} + \rho_1 \frac{d^{r-1}}{dx^{r-1}} + \cdots + \rho_{r-1} \frac{d}{dx} + \rho_r \equiv R$$

and

$$\sigma_0 \frac{d^s}{dx^s} + \sigma_1 \frac{d^{s-1}}{dx^{s-1}} + \cdots + \sigma_{s-1} \frac{d}{dx} + \sigma_s \equiv S,$$

our relation becomes

$$RAy - SBy \equiv 0.$$

By the lemma of this section this gives us at least one integral common to $Ay = 0$ and $By = 0$, because $s = a - n - 1$ is less than a even for the most unfavorable case, $n = 0$.

Therefore under all possibilities, the identical vanishing of $\Delta(x)$ is a sufficient condition that $Ay = 0$ and $By = 0$ shall have an integral not identically zero in common.

We also have the following

COROLLARY: If the n^{th} submatrix vanishes, then $Ay = 0$ and $By = 0$ have at least $(n + 1)$ independent integrals in common.

3. After defining *principal minors* of $\Delta(x)$, we shall take up the discussion of necessary and sufficient conditions for multiple common integrals.

DEFINITION. The k^{th} *principal minor* of $\Delta(x)$ is that determinant obtained by cutting off the first k and the last k rows and columns, and will be represented by $\Delta_k(x)$.

THEOREM. If $Ay = 0$ and $By = 0$ have $(m + 1)$ independent integrals in common, $\Delta(x)$ and its first m principal minors must vanish identically.

Take the system of equations obtained by cutting off the first i and also the last i equations of the system from which $\Delta(x)$ is obtained in §1, when arranged as there indicated. This will leave $(a + b - 2i)$ equations in which the equation of highest order is of order $(a + b - i - 1)$. Eliminating the $(a + b - 2i - 1)$ derivatives of y of highest order, we shall finally obtain a linear homogeneous differential equation in y which is at most of order i . The coefficients of the various derivatives of y in this equation are determinants of the i^{th} submatrix of $\Delta(x)$, each of which contains the first $(a + b - 2i - 1)$ columns of the i^{th} submatrix as columns. The coefficient of $\frac{d^i y}{dx^i}$ in this equation is $\Delta_i(y)$.

This new equation of order i will be satisfied by the $(m + 1)$ independent common integrals of $Ay = 0$ and $By = 0$. Therefore if $i \leq m$, every one of its coefficients must be zero; and so $\Delta_i(x)$ must vanish identically for $i = 1, 2, \dots, m$.

THEOREM. If $\Delta(x)$ and its first m principal minors vanish identically, $Ay = 0$ and $By = 0$ have at least $(m + 1)$ independent integrals in common.

We prove (1) that if $\Delta(x)$ and $\Delta_1(x)$ vanish identically, then $Ay = 0$ and $By = 0$ have at least two independent integrals in common. Then in order to use the method of complete induction, we prove (2) that if there are k independent common integrals when $\Delta(x)$ and $\Delta_i(x)$ [$i = 1, 2, \dots, (k - 1)$] vanish identically, there are at least $(k + 1)$ such integrals when $\Delta_k(x)$ also vanishes identically.

First. Let $\Delta(x)$ and $\Delta_1(x)$ vanish identically; further let the $(n_1 + 1)^{\text{st}}$ submatrix of $\Delta_1(x)$ be the first one which does not vanish. If now we multiply the elements of the various columns of the n_1^{th} submatrix by derivatives of y selected as indicated in the proof of the theorem of §2, and add together

in a properly chosen column as before, we shall obtain a determinant which must vanish identically. Since the elements of the last column of $\Delta(x)$ do not appear in $\Delta_1(x)$, we shall find that y does not appear as a term of this new determinant, although its derivatives do. Writing $b - n_1 - 2 = r_1$ and $a - n_1 - 2 = s_1$, the column containing the derivative of y can be written

$$\frac{d^{r_1}}{dx^{r_1}} Ay - A_{0,b-2} y, \frac{d^{r_1-1}}{dx^{r_1-1}} Ay - A_{0,b-3} y, \dots, \frac{d}{dx} Ay - A_{0,1} y, Ay - A_{0,0} y, \\ By - B_{00} y, \frac{d}{dx} By - B_{01} y, \dots, \frac{d^{s_1-1}}{dx^{s_1-1}} By - B_{0,a-3} y, \frac{d^{s_1}}{dx^{s_1}} By - B_{0,a-2} y,$$

where $A_{0,i}$, B_{0j} represent functions of x which are coefficients of y , or linear homogeneous differential operators of order zero. Expanding the determinant according to this column, we obtain a relation of the form

$$R_1 Ay - S_1 By - T_0 y \equiv 0,$$

where R_1 , S_1 , T_0 are linear homogeneous differential operators of orders r_1 , s_1 , and 0 respectively. Substituting a common integral Y , not identically zero, of $Ay = 0$ and $By = 0$ in this relation, we have $T_0 Y = 0$. Therefore T_0 vanishes identically and the above relation becomes

$$R_1 Ay - S_1 By \equiv 0.$$

Then, from the lemma, $Ay = 0$ and $By = 0$ must have at least $(n_1 + 2)$ and therefore at least two independent integrals in common.

Secondly. Let $\Delta(x)$ and $\Delta_i(x)$ ($i = 1, 2, \dots, k$) vanish identically; also let the $(n_k + 1)^{\text{st}}$ submatrix of $\Delta_k(x)$ be the first one which does not vanish.

As before, we multiply the elements of the various columns of the n_k^{th} submatrix of $\Delta_k(x)$ by the derivatives of y of proper order and add together in a properly selected column. There results a determinant which must vanish identically.

Since the elements of the last k columns of Δx do not enter $\Delta_k(x)$, y and its derivatives of order lower than k do not appear in this new determinant. Writing $(b - n_k - k - 1) = r_k$ and $(a - n_k - k - 1) = s_k$ the elements of this column can be written

$$\begin{aligned} \frac{d^{r_k}}{dx^{r_k}} Ay - A_{k-1, b-k-1} y, \frac{d^{r_k-1}}{dx^{r_k-1}} Ay - A_{k-1, b-k-2} y, \dots, \\ \frac{d}{dx} Ay - A_{k-1, 1} y, Ay - A_{k-1, 0} y, By - B_{k-1, 0} y, \frac{d}{dx} By - B_{k-1, 1} y, \dots, \\ \frac{d^{s_k-1}}{dx^{s_k-1}} By - B_{k-1, a-k-2} y, \frac{d^{s_k}}{dx^{s_k}} By - B_{k-1, a-k-1} y, \end{aligned}$$

where $A_{k-1, i}$, $B_{k-1, j}$ represent linear homogeneous differential operators of order $(k-1)$. Expanding according to this column we obtain a relation of the form

$$R_k Ay - S_k By - T_{k-1} y \equiv 0,$$

where R_k , S_k , T_{k-1} are linear homogeneous differential operators of order r_k , s_k , and $(k-1)$ respectively. Substituting k independent integrals $Y_1, Y_2, \dots, Y_k$ common to $Ay = 0$ and $By = 0$, which by hypothesis they must have on account of the identical vanishing of $\Delta(x)$ and $\Delta_i(x)$ [$i = 1, 2, \dots, (k-1)$], we find that the differential equation of $(k-1)^{\text{st}}$ order,

$$T_{k-1} y = 0,$$

has k independent integrals. Therefore $T_{k-1} y$ vanishes identically and our relation becomes

$$R_k Ay - S_k By \equiv 0.$$

From the lemma, $Ay = 0$ and $By = 0$ must have at least $(n_k + k + 1)$ and therefore at least $(k+1)$ independent integrals in common.

Our proof is now completed by the method of mathematical induction.

Combining the last two results we have the

THEOREM: *A necessary and sufficient condition that two linear homogeneous differential equations $Ay = 0$ and $By = 0$ shall have k independent integrals in common is that $\Delta(x)$ and its first $(k-1)$ principal minors shall vanish identically.*

This gives the

COROLLARY: *Two linear homogeneous differential equations $Ay = 0$ and $By = 0$ have k and only k independent integrals in common, if and only if $\Delta(x)$ and its first $(k-1)$ principal minors vanish identically and the k^{th} principal minor does not.*

4. I stated in the introduction that this discussion would give us all the results for the corresponding problem of algebra. I shall now justify that statement.

First, if we take the differential equation with constant coefficients and follow the process given for the necessity of the condition, we shall find that the resulting formula is just what we should obtain if we put λ^k for $\frac{d^k y}{dx^k}$ and $\lambda^0 = 1$ for y and then carried out the investigation as is ordinarily done in algebra.

Further we know that the solutions of linear homogeneous differential equations with constant coefficients are in general of the form $e^{\lambda x}$, where λ must satisfy the equation obtained above. From this consideration, we see clearly why we should obtain the solution of the algebraic problem as a special case from that of the differential equation.

5. The following particular cases are given by way of illustration.

1. Take

$$Ay \equiv \frac{d^3 y}{dx^3} + (x - a) \frac{d^2 y}{dx^2} + x(x - a) \frac{dy}{dx} - ax^2 y,$$

$$By \equiv \frac{d^2 y}{dx^2} + (x - a) \frac{dy}{dx} - axy.$$

It can easily be verified that if R and S be taken as follows:

$$R \equiv (1 - x^2) \frac{d}{dx} + x(3 - x^2),$$

$$S \equiv (1 - x^2) \frac{d^2}{dx^2} + x(3 - x^2) \frac{d}{dx} - (1 - x^2)^2,$$

then

$$RAy \equiv SBy.$$

Therefore $Ay = 0$ and $By = 0$ have at least one common integral not identically zero, since $a - s = 1$. By substituting e^{ax} for y in the two differential equations we find them to be satisfied.

2. Take

$$Ay \equiv \frac{d^3 y}{dx^3} + x(x + 1) \frac{d^2 y}{dx^2} + (x^3 + x^2 + 1) \frac{dy}{dx} + x(x^3 + 2)y,$$

$$By \equiv \frac{d^3 y}{dx^3} + 2x \frac{d^2 y}{dx^2} + (2x^2 + 1) \frac{dy}{dx} + x(x^2 + 2)y,$$

and

$$R \equiv x(x-1) \frac{d}{dx} + (x^3 - x^2 - 2x + 1),$$

$$S \equiv x(x-1) \frac{d}{dx} + (x^4 - x^3 - 2x + 1).$$

Then

$$RAY \equiv SBY.$$

From the lemma, we conclude that the two equations $AY = 0$ and $BY = 0$ have at least two independent integrals in common, since $a - s = 2$. In fact, the common integrals are the solutions of the differential equation

$$\frac{d^2y}{dx^2} + x \frac{dy}{dx} + x^2y = 0.$$

3. Take

$$AY \equiv \frac{d^4y}{dx^4} + x \frac{d^3y}{dx^3} + (x^2 - 1) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - x^2y,$$

$$BY \equiv \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} - \frac{dy}{dx} - x^2y,$$

and

$$R \equiv x(x^3 - 2x^2 + x - 1) \frac{d}{dx} + (x^6 - 2x^5 + 4x^4 - 7x^3 - 2x + 2),$$

$$S \equiv x(x^3 - 2x^2 + x - 1) \frac{d^2}{dx^2} + (x^5 + x^4 - 5x^3 - x^2 - 2x + 2) \frac{d}{dx} \\ + (-2x^6 + 2x^5 + 2x^4 - x^3 - x^2 + 3x).$$

We can very easily verify that $AY = 0$ and $BY = 0$ have e^x and e^{-x} as common solutions, *i. e.* the solutions of the differential equation

$$\frac{d^2y}{dx^2} - y = 0.$$

In these illustrations, we find that the number of independent integrals which the two equations have in common is exactly $(a - s)$. The reason for this is that the expressions given for R and S are those of the lowest order for which the identity is true.